Partition function for a one-dimensional δ-function Bose gas

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The N-particle partition function of a one-dimensional δ-function Bose gas is calculated explicitly using only the periodic boundary condition (the Bethe ansatz equation). The N-particles cluster integrals are shown to be the same as those by the thermal Bethe ansatz method.

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I. INTRODUCTION

Extending the work by Lieb and Liniger [1], Yang and Yang [2] presented an ingenious method to study the equilibrium thermodynamics of a one-dimensional system of bosons with repulsive δ-function interaction. The Hamiltonian of the system is

$$H_N = -\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial x_i^2} + \kappa \sum_{j\neq i} \delta(x_j - x_i)\right), \tag{1.1}$$

where κ is the coupling constant and is assumed to be positive. Throughout the paper, the Planck constant and the mass of a particle are chosen to be $\hbar = 1$ and $2m = 1$. This system is a first quantized version of the nonlinear Schrödinger model described by the Hamiltonian operator

$$H = \int dx[\phi_x^+ \phi_x + \kappa \phi^+ \phi^+ \phi\phi], \tag{1.2}$$

where $\phi(x,t)$ and $\phi^+(x,t)$ are boson field operators. The method invented by Yang and Yang is called the thermal Bethe ansatz (TBA) method since it starts from the Bethe ansatz wave function [3] and enables us to calculate the thermodynamic quantities at finite temperature. While there have been many successful applications to quantum particle and spin systems, not much study on the method itself has been done. A crucial assumption is the form of the entropy. Thacker [4] studied the δ-function Bose gas in infinite volume by the field-theoretic perturbation method and reproduced the results of the TBA method. One of the authors (M.W.) [5] presented a bosonic formulation of the TBA method and calculated the grand partition function of the system at infinite volume. In this paper, we calculate explicitly the N-particle partition function of the Hamiltonian (1.1) under the periodic boundary condition. That is, the assumptions of the TBA method are not used and all the calculations are carried out exactly at finite volume. A preliminary result for $N = 2, 3, 4$ has been reported in Ref. [6].

The outline of the paper is the following. In Sec. II, we present a method to evaluate the partition function for the N-particle system. A key idea is that we use only the periodic boundary condition of the wave function. In Sec. III, we calculate the N-particle cluster integrals using the partition function. An explicit calculation for the N-particle system has been done. The results are compared with those given by the TBA. We prove that both results completely agree. The last section is devoted to concluding remarks. Technical details of calculations are summarized in Appendixes A–D.

II. N-PARTICLE PARTITION FUNCTION

We study a statistical mechanics of the quantum N-particle system (1.1). Let E, L, and $\{k_i\}$ denote the total energy, the system size, and the wave numbers. It is known that the total energy and the wave numbers are determined by the following relations:

$$E = \sum_{i=1}^{N} k_i^2, \tag{2.1}$$

$$Lk_i = 2\pi n_i + \sum_{j<i} \Delta(k_j - k_i) - \sum_{j>i} \Delta(k_i - k_j), \tag{2.2}$$

where $\{n_i\}$ are integers satisfying the condition,

$$n_i \geqslant n_{i+1}, \tag{2.3}$$

and $\Delta(k)$ is the phase shift for two-body scattering,

$$\Delta(k) = -2\tan^{-1}\left(\frac{\kappa}{k}\right). \tag{2.4}$$

The relation (2.2) is obtained from the periodic boundary condition imposed on the N-particle eigenfunction and is called the Bethe ansatz equation. The range of the function (2.4) is assumed to be $(-2\pi, 0)$, and this phase shift is not a "true" one in the sense of Ref. [5]. That is, the phase-shift function (2.4) has analyticity on the real axis. The appearance of only the two-body S-matrix in (2.2) implies the factorization of S-matrices, which is one of the remarkable properties of integrable systems.

The partition function for the N-particle system is defined by

$$Z_N = \sum_{\{n_j\}} \exp\left(-\beta \sum_{i=1}^{N} k_i^2\right), \tag{2.5}$$

where, with T being the absolute temperature,

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$$\beta = \frac{1}{k_B T}, \tag{2.6}$$

and the summation is over all possible configurations of $\{n_j\}$ under the condition (2.3). Without recourse to the TBA method, we calculate the partition function Z_N only by using the relations (2.1)–(2.4). We explain a method in three steps.

A. Change of summation

In terms of new variable and function,

$$\tilde{n}_m \equiv n_m - m + \frac{N+1}{2}, \tag{2.7}$$

$$\tilde{\Delta}(k) \equiv \Delta(k) + \pi, \tag{2.8}$$

the periodic boundary condition (2.2) is changed into

$$Lk_i = 2\pi\tilde{n}_i + \sum_{j\neq i} \tilde{\Delta}(k_j - k_i). \tag{2.9}$$

We interpret (2.9) as analytic relations between real numbers $\{\tilde{n}_i\}$ and $\{k_i\}$. From the symmetry of (2.9), we see that, when $\{\tilde{n}_1,\ldots,\tilde{n}_i,\ldots,\tilde{n}_j,\ldots,\tilde{n}_N\}$ corresponds to $\{k_1,\ldots,k_i,\ldots,k_j,\ldots,k_N\}$, $\{\tilde{n}_1,\ldots,\tilde{n}_j,\ldots,\tilde{n}_i,\ldots,\tilde{n}_N\}$ should correspond to $\{k_1,\ldots,k_j,\ldots,k_i,\ldots,k_N\}$.

It is convenient to introduce the following set function:

$$\Theta(\sigma) \equiv \left\{ \theta \,\middle|\, \begin{array}{c} n\in\sigma, \quad \sigma'\in\theta, \quad n\in\sigma' \\ \sigma'',\sigma'''\in\theta, \quad \sigma''\neq\sigma''' \Rightarrow n''\in\sigma'', \quad n'''\in\sigma''', \quad n''\neq n''' \end{array} \right\}. \tag{2.10}$$

A domain σ of the function is a family of arbitrary finite sets. The first condition, $n\in\sigma, \sigma'\in\theta, n\in\sigma'$, means that the sum of the elements in θ is σ. The second condition, $\sigma'',\sigma''' \in\theta, \sigma''\neq\sigma'''\Rightarrow n''\in\sigma'', n'''\in\sigma''', n''\neq n'''$, means that the sum is a direct sum. That is, the image $\Theta(\sigma)$ is all the families of sets whose direct sum is the set σ. If the argument is an integer, we define

$$\Theta(N) \equiv \Theta(\{1,2,\ldots,N\}). \tag{2.11}$$

For example,

$$\Theta(9) \ni \{\{1,2\},\{3\},\{4\},\{5,6\},\{7,8\},\{9\}\} = \theta_9. \tag{2.12}$$

Figure 1 illustrates θ_9 in (2.12). And we define a symbol, for arbitrary function $f(n_1,\ldots,n_N)$,

$$\sum(\theta,\{n_i\}) f(n_1,\ldots,n_N) \equiv \sum_{\{n_\sigma\},\sigma\in\theta} f(n_1,\ldots,n_N), \quad \theta\in\Theta(N), \quad n_i = n_\sigma, i\in\sigma. \tag{2.13}$$

This symbol, $\Sigma(\theta,\{n_i\})f(n_i)$, indicates that a function f is summed up over all n_i on the condition that $n_j = n_k$ if an element which contains both j and k exists.

The following formula can be proved (Appendix A):

$$\sum_{n_1<\ldots<n_N} f(n_1,\ldots,n_N) = \frac{1}{N!}\sum_{\theta\in\Theta(N)} F(\theta) \sum(\theta,\{n_i\}) f(n_1,\ldots,n_N), \tag{2.14}$$

where

$$F(\theta) = \prod_{\sigma\in\theta} (-1)^{(M_\sigma - 1)}(M_\sigma - 1)!, \tag{2.15}$$

$$f(n_1,\ldots,n_i,\ldots,n_j,\ldots,n_N) = f(n_1,\ldots,n_j,\ldots,n_i,\ldots,n_N). \tag{2.16}$$

Here and hereafter, the number of elements in a set σ is denoted by M_σ.

By use of the formula (2.14), the partition function is rewritten as

$$Z_N = \sum_{\cdots\tilde{n}_i>\tilde{n}_{i+1}>\cdots} \exp\left(-\beta\sum_{j=1}^N k_j^2\right) = \frac{1}{N!}\sum_{\theta\in\Theta(N)} F(\theta)\sum(\theta,\{\tilde{n}_i\})\exp\left(-\beta\sum_{j=1}^N k_j^2\right). \tag{2.17}$$

B. Replacement of summations by integrals

To evaluate (2.17), we have a useful formula,

$$\sum_{n=\text{integer}} f(n) = \sum_{n=\text{integer}} \int_{-\infty}^{\infty} dn'\, f(n')\exp(-2\pi i n n') \quad \text{if } |f(n)| < \exp(-rn^2), \quad r>0. \tag{2.18}$$

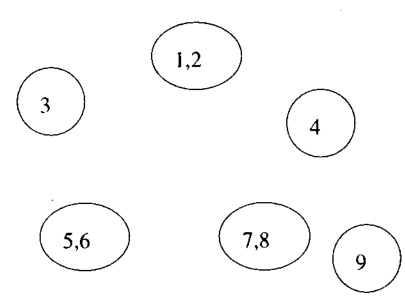

FIG. 1. A graphical representation of a set θ_9 in (2.12).

This can be proved by the Fourier transform and the Jacobi's imaginary transformation for elliptic theta functions. Applying the formula (2.18), we replace summations in (2.17) by integrals,

$$Z_N=\frac{1}{N!}\sum_{\theta\in\Theta(N)}F(\theta)\sum_{\{n_\sigma\},\sigma\in\theta}(-1)^{(N-1)\Sigma_{\sigma\in\theta}n_\sigma}\int\prod_{\sigma\in\theta}d\tilde{n}'_\sigma\exp\left(\sum_{\sigma\in\theta}(-\beta k_\theta^2-2\pi i n_\sigma\tilde{n}'_\sigma)\right). \tag{2.19}$$

To proceed further, we define four symbols.

First,

$$\Lambda(\theta)\equiv\left\{\lambda\left|\begin{array}{c}\lambda=\{\xi|\xi=\{\sigma,\sigma'\}\quad\sigma,\sigma'\in\theta\}\\ \lambda'\subseteq\lambda\,\text{the number of elements in }\{\sigma|\xi\in\lambda'\quad\sigma\in\xi\}>M_{\lambda'}\end{array}\right.\right\}, \tag{2.20}$$

$$\theta=\{\sigma_1,\sigma_2,\ldots\}.$$

The set function $\Lambda(\theta)$ consists of elements λ. λ contains elements ξ, each of which has two elements σ, σ'. We call ξ connection between the two elements σ, σ'. The connectivity is referred to as the pattern of connections. $\Lambda(\theta)$ represents a set of all the patterns of connections that have no ring of the connections. In Fig. 2, the connection and ring of connections are illustrated:

$$\Lambda(\theta_9)\ni\{\{\{1,2\},\{3\}\},\{\{1,2\},\{4\}\},\{\{5,6\},\{7,8\}\}\}=\lambda_9. \tag{2.21}$$

The pattern of connection λ_9 is shown in Fig. 3. As a special case, $\Lambda(\theta)$ contains a pattern of no connection, i.e., $\Lambda(\theta)\ni\emptyset$.

Second,

$$n(\sigma,\lambda)\equiv\text{the number of elements in }\{\xi|\xi=\{\sigma,\sigma'\},\ \sigma'\in\{\sigma_1,\sigma_2,\ldots\},\ \xi\in\lambda\}, \tag{2.22}$$

$$\sigma\in\{\sigma_1,\sigma_2,\ldots\},\quad\lambda\in\Lambda(\{\sigma_1,\sigma_2,\ldots\}).$$

In words, $n(\sigma,\lambda)$ is a number of elements which are linked with σ.

Third,

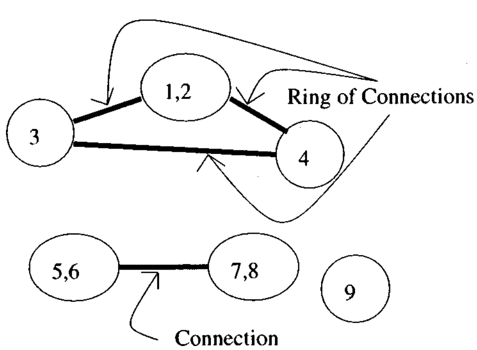

FIG. 2. The connection and the ring of connection.

$$G(\lambda)\equiv\{\lambda'\subseteq\lambda|\{\sigma,\sigma'\}\in\lambda,\ \{\sigma,\sigma''\}\in\lambda'\Rightarrow\{\sigma,\sigma''\}\in\lambda'\},$$

$$\lambda\in\Lambda(\{\sigma_1,\sigma_2,\ldots\}). \tag{2.23}$$

That is, $G(\lambda)$ means a set of connection patterns each of which is a cluster of the connection pattern λ. Here, the cluster means that if two connections ξ' and ξ'' in λ are linked with a common element σ, then ξ' and ξ'' are in the same cluster. For example,

$$G(\lambda_9)\ni\{\{\{1,2\},\{3\}\},\{\{1,2\},\{4\}\}\}. \tag{2.24}$$

A pattern of no connection is included in $G(\lambda)$.

Fourth,

$$g([\lambda,\theta])\equiv\left\{\sigma\left|\begin{array}{c}\sigma\in\theta,\quad\xi\in\lambda,\quad\sigma\ni\xi\\ \lambda'\in G(\lambda),\quad\lambda'\neq\emptyset,\quad\sigma=\bigcup_{\{\sigma',\sigma''\}\in\lambda'}(\sigma'\cup\sigma'')\end{array}\right.\right\},$$

$$\theta=\{\sigma_1,\sigma_2,\ldots\},\quad\lambda\in\Lambda(\theta). \tag{2.25}$$

$[\lambda,\theta]$ means that λ is consistent with θ; an element which is linked with λ is in θ. $g([\lambda,\theta])$ indicates a set of direct sum "elements of θ" which are connected by one cluster in the connection pattern λ. The reason why an element is quoted here is that the element, at the same time, is a set. If an element of θ has no connection, the "element" belongs to $g([\lambda,\theta])$. For example,

$$g([\lambda_9,\theta_9])=\{\{1,2,3,4\},\{5,6,7,8\},\{9\}\}. \tag{2.26}$$

The following formula is proved (see a proof in Appendix B):

$$|A_{\sigma,\sigma'}(\theta)|=\sum_{\lambda\in\Lambda(\theta)}\left(I([\lambda,\theta])\prod_{\{\sigma'',\sigma'''\}\in\lambda}x_{\sigma'',\sigma'''}\right), \tag{2.27}$$

with

$$I([\lambda,\theta])\equiv L^{N-M_\lambda}\left(\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)-1}\right)\left(\prod_{\sigma'\in g([\lambda,\theta])}M_{\sigma'}\right), \tag{2.28}$$

$$A_{\sigma,\sigma'}(\theta)=\begin{cases}L+\sum_{\sigma''\in\theta}M_{\sigma''}x_{\sigma,\sigma''} & \text{if } \sigma=\sigma',\\ -M_{\sigma'}x_{\sigma,\sigma'} & \text{if } \sigma\neq\sigma'\end{cases}, \tag{2.29}$$

$$\theta\in\Theta(N),\quad \sigma,\sigma'\in\theta,\quad x_{\sigma,\sigma'}\equiv x_{\sigma',\sigma},\quad x_{\sigma,\sigma}\equiv 0.$$

Using the formula (2.27), we change variables in (2.19) from $\{\tilde{n}_i\}$ to $\{k_i\}$. In terms of the above introduced symbols, we obtain

$$Z_N=\frac{1}{N!}\sum_{\theta\in\Theta(N)}F(\theta)\sum_{\{n_\sigma\},\sigma\in\theta}(-1)^{(N-1)\Sigma_{\sigma\in\theta}n_\sigma}\times\int\prod_{\sigma\in\theta}\frac{dk_\sigma}{2\pi}\sum_{\lambda\in\Lambda(\theta)}I([\lambda,\theta])\times\left(\prod_{\{\sigma,\sigma'\}\in\lambda}\frac{d\Delta}{dk}(k_\sigma-k_{\sigma'})\right)\exp\left[\sum_{\sigma\in\theta}\left\{-\beta M_\sigma k_\sigma^2-in_\sigma\left(Lk_\sigma-\sum_{\sigma'\in\theta,\neq\sigma}M_{\sigma'}\tilde{\Delta}(k_{\sigma'}-k_\sigma)\right)\right\}\right]. \tag{2.30}$$

$F(\theta)$ is defined in (2.15). Note that because of a relation, if $\tilde{n}_i=\tilde{n}_j$, then $k_i=k_j$, the Jacobian of the transformation can be written explicitly.

C. Change of integral paths

We change the integral paths in (2.30) from $\{(-\infty,\infty)\}$ to $\{(-\infty-(Ln_\sigma/2\beta M_\sigma)i,\infty-(Ln_\sigma/2\beta M_\sigma)i)\}$. In what follows, ''an integral path steps over a residue,'' means that when an integral path is moved, there is a residue in the region surrounded by the initial and final integral paths. It is important to note that each integral path in (2.30) does not step over any residue by this change of the integral path.

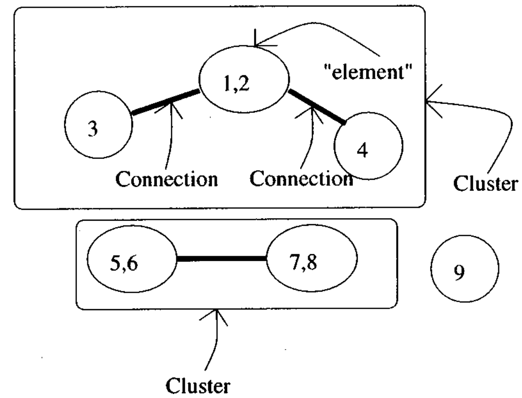

FIG. 3. The pattern of connection $[\lambda_9,\theta_9]$ in (2.21). Here, $[\lambda_9,\theta_9]$ indicates that λ_9 is consistent with θ_9.

Performing the change of integral paths as introduced above, we arrive at an expression of the partition function,

$$Z_N=\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda(\theta)}J([\lambda,\theta])S([\lambda,\theta]) \tag{2.31}$$

with

$$J([\lambda,\theta])\equiv\frac{L^{N'-M_\lambda}}{N'!}\left(\prod_{\sigma'\in g([\lambda,\theta])}M_{\sigma'}\right)\times\left(\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)-1}(-1)^{(M_\sigma-1)}(M_\sigma-1)!\right), \tag{2.32}$$

$$S([\lambda,\theta])\equiv\sum_{\{n_\sigma\},\sigma\in\theta}(-1)^{(N'-1)\Sigma_{\sigma\in\theta}n_\sigma}\exp\left(-\frac{L^2}{4\beta}\sum_{\sigma\in\theta}\frac{n_\sigma^2}{M_\sigma}\right)\int\prod_{\sigma\in\theta}\frac{dk_\sigma}{2\pi}\left(\prod_{\{\sigma,\sigma'\}\in\lambda}\frac{d\Delta}{dk}\left(k_\sigma-k_{\sigma'}-\frac{Ln_\sigma}{2\beta M_\sigma}i+\frac{Ln_{\sigma'}}{2\beta M_{\sigma'}}i\right)\right)\times\exp\left[\sum_{\sigma\in\theta}\left\{-\beta M_\sigma k_\sigma^2+in_\sigma\Sigma_{\sigma'\in\theta,\neq\sigma}M_\sigma\tilde{\Delta}\left(k_{\sigma'}-k_\sigma-\frac{Ln_\sigma}{2\beta M_\sigma}i+\frac{Ln_{\sigma'}}{2\beta M_{\sigma'}}i\right)\right\}\right], \tag{2.33}$$

$$N'\equiv\sum_{\sigma\in\theta}M_\sigma. \tag{2.34}$$

This formula enables us to calculate the partition function in any order of the system size L. In another way of writing, we can calculate the partition function for the system with finite size as

$$Z_N=\sum_n U_n(L,\beta)\exp\left(-\frac{nL^2}{4N\beta}\right), \tag{2.35}$$

$$\lim_{L\to\infty}\frac{U_n(L,\beta)}{L^N}<A(\beta),\quad n\geqslant 0.$$

It should be remarked that U_n for arbitrary n is explicitly obtained from (2.31)–(2.34).

III. CLUSTER EXPANSION

The cluster expansion for the equation of state is defined by

$$p\beta=\lim_{L\to\infty}L^{-1}\log\left(\sum_{N=0}^{\infty}(z^N Z_N)\right) \tag{3.1}$$

$$=\sum_{N=1}^{\infty}z^N b_N, \tag{3.2}$$

where z is the fugacity (rigorously, the absolute activity),

$$z=\exp(\beta\mu). \tag{3.3}$$

The partition function $\{Z_N\}$ and the cluster integral $\{b_N\}$ are related as

$$b_N=\lim_{L\to\infty}\sum_{\theta\in\Theta(N)}\frac{(M_\theta-1)!(-1)^{M_\theta-1}\prod_{\sigma\in\theta}M_\sigma!}{LN!}\prod_{\sigma\in\theta}Z_{M_\sigma}. \tag{3.4}$$

Expanding the right-hand side of Eq. (3.1) in powers of z and combinatorially summing up the coefficients, we can prove the relation (3.4).

We define that

$$\Lambda_c(\theta)=\{\lambda|\lambda\in\Lambda(\theta)\quad M_{g(\lambda,\theta)}=1\}. \tag{3.5}$$

$\Lambda_c(\theta)$ is a set of connection patterns which consist of one cluster (this cluster has nothing to do with the cluster integral).

Substituting the expression (2.31) of Z_N into (3.4), we obtain

$$b_N=\frac{1}{L}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}J([\lambda,\theta])S_\infty([\lambda,\theta]), \tag{3.6}$$

where $J(\lambda,\theta)$ is defined in (2.32), and

$$S_\infty([\lambda,\theta])\equiv\int\prod_{\sigma\in\theta}\frac{dk_\sigma}{2\pi}\left(\prod_{\{\sigma,\sigma'\}\in\lambda}\frac{d\Delta}{dk}(k_\sigma-k_{\sigma'})\right)\times\exp\left(\sum_{\sigma\in\theta}(-\beta M_\sigma k_\sigma^2)\right). \tag{3.7}$$

This is proved in Appendix C.

To compare the results in this paper with those in Ref. [5], we define a function $K(k)$ by

$$\frac{d\Delta(k)}{dk}-2\pi\delta(k)\equiv K(k). \tag{3.8}$$

Note that for the noninteracting case, $K(k)\equiv0$.

In term of $K(k)$, the cluster integral b_N can be expressed as follows:

$$b_N=\frac{1}{L}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}J'([\lambda,\theta])S'_\infty([\lambda,\theta]), \tag{3.9}$$

with

$$J'([\lambda,\theta])\equiv\frac{L}{(N-1)!}\left(\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)-1}(M_\sigma-1)!\right), \tag{3.10}$$

$$S'_\infty([\lambda,\theta])\equiv\int\prod_{\sigma\in\theta}\frac{dk_\sigma}{2\pi}\left(\prod_{\{\sigma,\sigma'\}\in\lambda}K(k_\sigma-k_{\sigma'})\right)\times\exp\left(\sum_{\sigma\in\theta}(-\beta M_\sigma k_\sigma^2)\right). \tag{3.11}$$

A proof of Eq. (3.9) is given also in Appendix C.

The cluster integrals (3.9) agree with those derived by the TBA method (see Appendix D for detail calculation). In this way, we have proved that the thermal Bethe ansatz (TBA) method by Yang and Yang gives the exact equation of state.

IV. CONCLUDING REMARKS

Taking a one-dimensional δ-function Bose gas as a typical example of integrable systems, we have derived the N-particle partition function. A method in this paper, referred to as the direct method, is an exact analysis of the partition function only based on the periodic boundary condition. Using the explicit expression of the partition function, we have calculated the N-particle cluster integral, and proved a perfect agreement between the results of this direct method and the thermal Bethe ansatz (TBA) method.

The extensions and applications of the direct method to integrable and nonintegrable systems may clarify mathematical structures of the TBA method. Those problems are left for future studies.

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APPENDIX A: A PROOF ON (2.14)

We prove (2.14), that is,

$$\sum_{n_l<\cdots<n_N}f(n_1,\ldots,n_N)=\frac{1}{N!}\sum_{\theta\in\Theta(N)}F(\theta)\sum(\theta,\{n_i\})\times f(n_1,\ldots,n_N), \tag{A1}$$

where $F(\theta)$ and $\Sigma(\theta,\{n_i\})$ are defined in (2.15) and (2.14), and $f(n_1,\ldots,n_i,\ldots,n_j,\ldots,n_N)$ satisfies

$$f(n_1,\ldots,n_i,\ldots,n_j,\ldots,n_N)=f(n_1,\ldots,n_j,\ldots,n_i,\ldots,n_N). \tag{A2}$$

We define a semiorder on a set $\theta\in\Theta(N)$,

$$\theta'\leqslant\theta\overset{\text{def}}{\Leftrightarrow}\sigma'\in\theta',\quad\sigma\in\theta,\quad\sigma'\subseteq\sigma. \tag{A3}$$

A sufficient condition of Eq. (A1) is

$$\sum_{\theta' \leq \theta} F(\theta') = \delta_{\theta,\theta_N}, \quad \theta \in \Theta(N), \tag{A4}$$

where

$$\theta_N \equiv \{\{1\},\{2\},\ldots,\{N\}\}, \tag{A5}$$

$$\delta_{\theta,\theta_N} \equiv \begin{cases} 0 & \text{if } \theta = \theta_N, \\ 1 & \text{otherwise.} \end{cases} \tag{A6}$$

We consider a function,

$$X(\{x_{i,j}\}) \equiv \prod_{j=2}^{N} \left(1 - \sum_{i=1}^{j-1} x_{i,j} \right) \tag{A7}$$

and a mapping P,

$$P: h(\{x_{i,j}\}) \to \sum_i n_i \theta_i, \quad \theta_i \in \Theta(N) \tag{A8}$$

which satisfies the following relations:

$$P(h_1 + h_2) = P(h_1) + P(h_2),$$
$$P(nh) = nP(h), \tag{A9}$$
$$P\left(\prod_i x_{n_i,m_i} \right) = g([\{\xi | \xi = \{\{n_i\},\{m_i\}\}\}, \theta_N]),$$

where h, h_1, h_2 are arbitrary polynomial functions, and $g([\lambda,\theta])$ is defined in (2.25). It is readily shown that the following relation holds:

$$P(X) = \sum_{\theta \in \Theta(N)} F(\theta)\theta. \tag{A10}$$

If we substitute

$$x_{i,j,\theta} \equiv \begin{cases} 1 & \text{if } \theta \text{ and } n,m \in \sigma \\ 0 & \text{otherwise,} \end{cases} \tag{A11}$$

for $x_{i,j}$, the relation (A10) becomes

$$X(x_{ij} = x_{i,j,\theta}) = P(X(x_{i,j} = x_{i,j,\theta}))$$
$$= \sum_{\theta' \leq \theta} F(\theta') = \delta_{\theta,\theta_N}. \tag{A12}$$

Equation (A12) is the sufficient condition (A4) of Eq. (A1). Thus, Eq. (2.14) is proved.

APPENDIX B: A PROOF ON (2.27)

We prove (2.27), that is,

$$|A_{\sigma,\sigma'}(\theta)| = \sum_{\lambda \in \Lambda(\theta)} \left(I([\lambda,\theta]) \prod_{\{\sigma'',\sigma'''\} \in \lambda} x_{\sigma'',\sigma'''} \right), \tag{B1}$$

where $\theta \in \Theta(N)$, $\sigma, \sigma' \in \theta$, $x_{\sigma,\sigma'} \equiv x_{\sigma',\sigma}$, $x_{\sigma,\sigma} \equiv 0$, and $I([\lambda,\theta])$ and $A_{\sigma,\sigma'}(\theta)$ are defined in (2.28) and (2.29).

Only the terms $A_{\sigma_1,\sigma_1}(\theta)$, $A_{\sigma_1,\sigma_2}(\theta)$, $A_{\sigma_2,\sigma_1}(\theta)$, and $A_{\sigma_2,\sigma_2}(\theta)$ contain the variable x_{σ_1,σ_2}, and the minor determinant becomes

$$\begin{vmatrix} A_{\sigma_1,\sigma_1}(\theta) & A_{\sigma_1,\sigma_2}(\theta) \\ A_{\sigma_2,\sigma_1}(\theta) & A_{\sigma_2,\sigma_2}(\theta) \end{vmatrix} = (\alpha_{\sigma_1} + \alpha_{\sigma_2}) x_{\sigma_1,\sigma_2} + \alpha_{\sigma_1}\alpha_{\sigma_2}, \tag{B2}$$

where

$$\alpha_\sigma = L + \sum_{\sigma' \in \theta, \neq \sigma_1,\sigma_2} M_{\sigma'} x_{\sigma,\sigma'}. \tag{B3}$$

The right-hand side of (B2) contains only the terms of x_{σ_1,σ_2} and 1 when we regard the equation as a polynomial of x_{σ_1,σ_2}. Therefore, an exponent of a variable x_{σ_1,σ_2} in the determinant (B1) is 1 or 0.

We define a set,

$$\Lambda'(\theta) \equiv \{\lambda | \lambda = \{\xi | \xi = \{\sigma,\sigma'\} \;\; \sigma,\sigma' \in \theta\}\}, \quad \theta \in \{\sigma\}. \tag{B4}$$

While $\Lambda(\theta)$ (2.20) does not include λ which contains rings of connections, $\Lambda'(\theta)$ does include such kinds of λ. In terms of this set, the determinant is written as

$$|A_{\sigma,\sigma'}(\theta)| = \sum_{\lambda \in \Lambda'(\theta)} \left(I'([\lambda,\theta]) \prod_{\{\sigma'',\sigma'''\} \in \lambda} x_{\sigma'',\sigma'''} \right). \tag{B5}$$

Here $I'([\lambda,\theta])$ is an undetermined function which is projected on a polynomial of L.

We make an assumption: $I'([\lambda,\theta])$ is not zero on the condition of λ that there exists λ_r such that

$$\lambda_r \subseteq \lambda \quad \text{and} \quad \text{the number of elements in}$$
$$\{\sigma | \xi \in \lambda_r, \;\; \sigma \in \xi\} \leq M_{\lambda_r}. \tag{B6}$$

This assures that the connection pattern λ contains the rings of connections. We define a set

$$\overline{\lambda_r} \equiv \lambda - \lambda_r. \tag{B7}$$

We can choose λ_r such that

$$\xi \in \lambda, \;\; \xi_r \in \lambda_r \Rightarrow \xi \cap \xi_r = \emptyset. \tag{B8}$$

This means that λ_r consists of some cluster of λ, and λ_r contains some rings of connections. In this case, it is clear that

$$I'([\lambda,\theta]) = I'([\lambda_r,\theta]) I'([\overline{\lambda_r},\theta]) L^{-N}, \tag{B9}$$

and that

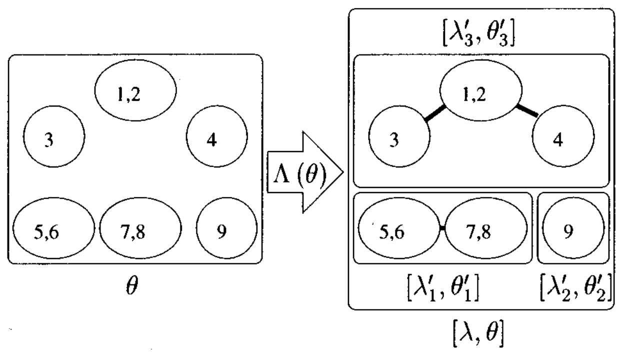

FIG. 4. A graphical representation of one of the patterns which are summed up in the left-hand side of Eq. (C7).

$$|A'_{\sigma,\sigma'}(\theta,\theta')| = I'([\lambda_r,\theta]) \prod_{\{\sigma'',\sigma'''\}\in\lambda} x_{\sigma'',\sigma'''} + \sum_{\lambda\in\Lambda'(\theta),\neq\lambda_r} \left(I''([\lambda,\theta]) \prod_{\{\sigma'',\sigma'''\}\in\lambda} x_{\sigma'',\sigma'''} \right), \tag{B10}$$

where

$$A'_{\sigma,\sigma'}(\theta,\theta') = \begin{cases} L & \text{if } \sigma,\sigma' \notin \theta' \, \sigma=\sigma' \\ \sum_{\sigma''\in\theta'} M_{\sigma''} x_{\sigma,\sigma''} & \text{if } \sigma,\sigma' \in \theta' \, \sigma=\sigma', \\ -M_{\sigma'} x_{\sigma,\sigma'} & \text{if } \sigma,\sigma' \in \theta' \, \sigma\neq\sigma', \\ 0 & \text{otherwise,} \end{cases} \tag{B11}$$

$$\theta' = \{\sigma | \sigma\in\xi, \quad \xi\in\lambda_r\}, \quad \sigma,\sigma'\in\theta.$$

$I''([\lambda,\theta])$ is an undetermined function. From (B11), we can see that

$$\sum_{\sigma\in\theta'} A'_{\sigma,\sigma'}(\theta,\theta') = 0 \tag{B12}$$

which indicates that row vectors of the matrix are linearly dependent; $|A'_{\sigma,\sigma'}(\theta,\theta')|$ is identically zero. This negates the assumption that $I'([\lambda,\theta])$ is not zero on the condition (B6). Therefore, we have

$$|A_{\sigma,\sigma'}(\theta)| = \sum_{\lambda\in\Lambda(\theta)} \left(I'([\lambda,\theta]) \prod_{\{\sigma,\sigma'\}\in\lambda} x_{\sigma,\sigma'} \right). \tag{B13}$$

A difference between (B5) and (B13) is the region of summation over λ. We notice that each term on the right-hand side is not made from off-diagonal elements of $A_{\sigma,\sigma'}(\theta)$; all the terms which are made from off-diagonal elements are canceled by diagonal elements. Then, we have

$$|A''_{\sigma,\sigma'}(\theta)| = \sum_{\lambda\in\Lambda(\theta)} \left(I'([\lambda,\theta]) \prod_{\{\sigma'',\sigma'''\}\in\lambda} x_{\sigma'',\sigma'''} \right) + \sum_{\lambda\in\Lambda'(\theta)} \sum_{\sigma_1,\sigma_2\in\theta} I^{(3)}([\lambda,\theta],\sigma_1,\sigma_2) \times x^2_{\sigma_1,\sigma_2} \prod_{\{\sigma_3,\sigma_4\}\in\lambda} x_{\sigma_3,\sigma_4} + \sum_{\lambda\in\Lambda'(\theta),\notin\Lambda(\theta)} \left(I^{(4)}([\lambda,\theta]) \times \prod_{\{\sigma'',\sigma'''\}\in\lambda} x_{\sigma'',\sigma'''} \right), \tag{B14}$$

where

$$A''_{\sigma,\sigma'}(\theta) = \begin{cases} L + \sum_{\sigma''\in\theta} M_{\sigma''} x_{\sigma,\sigma''} & \text{if } \sigma=\sigma', \\ 0 & \text{if } \sigma\neq\sigma', \end{cases}$$

and $I^{(3)}([\lambda',\theta],\sigma_1,\sigma_2)$ and $I^{(4)}([\lambda',\theta])$ are undetermined functions. Therefore, we obtain

$$I'([\lambda,\theta]) \equiv L^{N-M_\lambda} \left(\prod_{\sigma\in\theta} M_\sigma^{n(\sigma,\lambda)-1} \right) \left(\prod_{\sigma'\in g([\lambda,\theta])} M_{\sigma'} \right) = I([\lambda,\theta]), \tag{B15}$$

and (B13) with (B15) proves (B1).

APPENDIX C: A DERIVATION OF THE CLUSTER INTEGRALS (3.6) AND (3.9)

First, we prove (3.6). Substituting the expression of Z_N in (2.31) into (3.4) yields

$$b_N = \lim_{L\to\infty} \sum_{\theta\in\Theta(N)} \frac{(M_\theta - 1)!(-1)^{M_\theta - 1} \prod_{\sigma\in\theta} M_\sigma!}{LN!} \times \prod_{\sigma\in\theta} \sum_{\theta'\in\Theta(M_\sigma)} \sum_{\lambda'\in\Lambda(\theta')} J([\lambda',\theta']) S([\lambda',\theta']). \tag{C1}$$

$J([\lambda,\theta])$ and $S([\lambda,\theta])$ are defined in (2.32) and (2.33). Eliminating exponentially decreasing terms with respect to L, we have

$$b_N = \lim_{L\to\infty} \sum_{\theta\in\Theta(N)} \frac{(M_\theta - 1)!(-1)^{M_\theta - 1} \prod_{\sigma\in\theta} M_\sigma!}{LN!} \times \prod_{\sigma\in\theta} \sum_{\theta'\in\Theta(\sigma)} \sum_{\lambda'\in\Lambda(\theta')} J([\lambda',\theta']) S_\infty([\lambda',\theta']). \tag{C2}$$

The integral $S_\infty(\lambda,\theta)$ is defined in (3.7).

We define

$$\tilde{G}([\lambda,\theta]) \equiv \left\{ [\lambda',\theta'] \,\middle|\, \begin{array}{c} \lambda'=\emptyset,\;\; M_{\theta'}=1,\;\; \theta'\subset\theta,\;\; \xi\in\lambda,\;\; \sigma\notin\xi \\ \text{or} \\ \lambda'\in G(\lambda),\;\; \lambda'\neq\emptyset,\;\; \sigma\in\theta',\;\; \xi\in\lambda',\;\; \sigma\in\xi \end{array} \right\}. \tag{C3}$$

In words, $\tilde{G}([\lambda,\theta])$ is a set of elements each of which consists of connection pattern λ' and a set θ'. Here, λ' is a cluster of the connection pattern λ, and θ' is a subset of θ and is a set of elements that are linked with connections in λ'. Note that $\tilde{G}([\lambda,\theta])$ contains $[\emptyset,\{\sigma\}]$ when σ is not connected by λ.

From the definition, it is shown that the integral $S_\infty([\lambda,\theta])$ is factorized into ''connected'' integrals as

$$S_\infty([\lambda,\theta]) = \prod_{[\lambda',\theta']\in\tilde{G}([\lambda,\theta])} S_\infty([\lambda',\theta']). \tag{C4}$$

Similarly, the coefficient $J([\lambda,\theta])$ is factorized,

$$\left(\sum_{\sigma\in\theta} M_\sigma\right)! J([\lambda,\theta]) = \prod_{[\lambda'\theta']\in\tilde{G}[(\lambda,\theta])} \left(\sum_{\sigma'\in\theta'} M_{\sigma'}\right)! J([\lambda',\theta']). \tag{C5}$$

Due to (C4) and (C5), the cluster integral (C2) can be rewritten as

$$b_N = \lim_{L\to\infty} \sum_{\theta\in\Theta(N)} \frac{(M_\theta-1)!(-1)^{M_\theta-1}}{LN!} \prod_{\sigma\in\theta} \sum_{\theta'\in\Theta(\sigma)} \sum_{\lambda'\in\Lambda(\theta')} \prod_{[\lambda'',\theta'']\in\tilde{G}([\lambda',\theta'])} \left(\sum_{\sigma''\in\theta''} M_{\sigma''}\right)! J([\lambda'',\theta''])S_\infty([\lambda'',\theta'']). \tag{C6}$$

We can show that for arbitrary function $f([\lambda,\theta])$,

$$\sum_{\theta\in\Theta(N)} \sum_{\lambda\in\Lambda(\theta)} \prod_{[\lambda',\theta']\in\tilde{G}([\lambda,\theta])} f([\lambda',\theta']) = \sum_{\theta''\in\Theta(N)} \prod_{\sigma''\in\theta''} \sum_{\theta'\in\Theta(\sigma'')} \sum_{\lambda'\in\Lambda_c(\theta')} f([\lambda',\theta']). \tag{C7}$$

The left-hand side of (C7) is a summation over all the patterns which are generated by the following process: first divide a set $\{1,...,N\}$ into elements of θ, then connect them with λ. The right-hand side of (C7) is a summation over all the patterns which generated by the following process: first divide a set $\{1,...,N\}$ into sets of θ'' each of which is a direct sum of elements connected by a cluster λ', second divide each of the sets into elements of θ', then define connection pattern λ', which consists of one cluster, of elements in θ'. Figures 4 and 5 illustrate graphical representations of both sides of patterns in (C7).

Due to (C7), the cluster integral (C6) becomes

$$b_N = \lim_{L\to\infty} \sum_{\theta\in\Theta(N)} \frac{(M_\theta-1)!(-1)^{M_\theta-1}}{LN!} \times \prod_{\sigma\in\theta} \sum_{\theta'\in\Theta(\sigma)} \prod_{\sigma'\in\theta'} M_{\sigma'}! \times \sum_{\theta''\in\Theta(\sigma')} \sum_{\lambda''\in\Lambda_c(\theta'')} J([\lambda'',\theta''])S_\infty([\lambda'',\theta'']). \tag{C8}$$

We define a family of sets $\tilde{g}(\tau)$,

$$\tilde{g}(\tau) = \left\{ \sigma \,\middle|\, \sigma = \bigcup_{\sigma'\in\theta'} \sigma' \quad \theta'\in\tau \right\}, \tag{C9}$$

where an element of τ is also a family of sets. To repeat, an element of $\tilde{g}(\tau)$ is a sum of sets of which a family of sets, an element of τ, consists. For example,

$$\tilde{g}(\{\{\{1,2\},\{3\}\},\{\{4,5\},\{6,7\}\}\}) = \{\{1,2,3\}\{4,5,6,7\}\}. \tag{C10}$$

We can show that for arbitrary functions $f_1(\theta), f_2(\sigma)$,

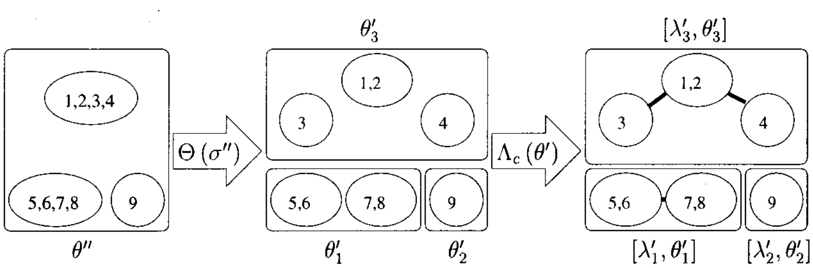

FIG. 5. A graphical representation of one of the patterns which are summed up in the right-hand side of Eq. (C7).

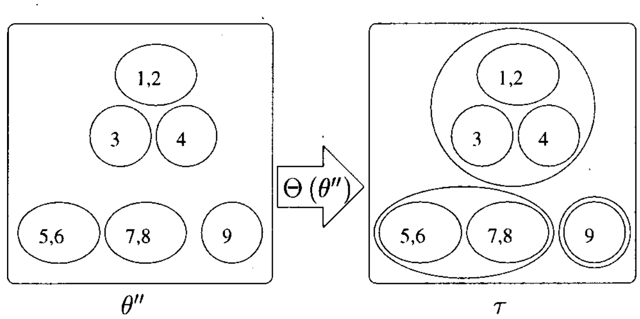

FIG. 6. A graphical representation of one of the patterns which are summed up in the left-hand side of Eq. (C11).

$$\sum_{\theta''\in\Theta(N)}\left(\prod_{\sigma'\in\theta''}f_2(\sigma')\right)\sum_{\tau\in\Theta(\theta'')}f_1(\tilde{g}(\tau))$$
$$=\sum_{\theta\in\Theta(N)}f_1(\theta)\prod_{\sigma\in\theta}\sum_{\theta'\in\Theta(\sigma)}\prod_{\sigma'\in\theta'}f_2(\sigma'). \tag{C11}$$

In the left-hand side of (C11), one sums up all the patterns which are generated by the following process: first divide $\{1,\ldots,N\}$ into elements in θ'', then make family of sets τ. Each of the sets consist of some elements in θ'', and $\tilde{g}(\tau)=\theta$. In the right-hand side of (C11), one sums up all the patterns which are generated by the following process: first divide a set $\{1,\ldots,N\}$ into elements of θ, then divide each of the elements in θ into elements of θ'. Figures 6 and 7 illustrate graphical representations of both sides of patterns in (C11).

Using a formula (C11) in (C6), b_N becomes

$$b_N=\lim_{L\to\infty}\sum_{\theta\in\Theta(N)}\left(\prod_{\sigma\in\theta}M_\sigma!\sum_{\theta''\in\Theta(\sigma)}\sum_{\lambda''\in\Lambda_c(\theta'')}J([\lambda'',\theta''])\right.$$
$$\left.\times S_\infty([\lambda'',\theta''])\right)\sum_{\tau\in\Theta(\theta)}\frac{(M_\tau-1)!(-1)^{M_\tau-1}}{LN!}. \tag{C12}$$

Note that we have used a relation $M_{\tilde{g}(\tau)}=M_\tau$.

Substitutions of $L^N/N!$ into Z_N in (3.4) and (3.1) give a relation

$$\sum_{\theta\in\Theta(N)}(M_\theta-1)!(-1)^{M_\theta-1}=\delta_{N,1}. \tag{C13}$$

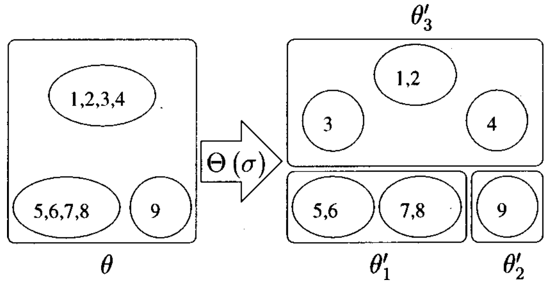

FIG. 7. A graphical representation of one of the patterns which are summed up in the right-hand side of Eq. (C11).

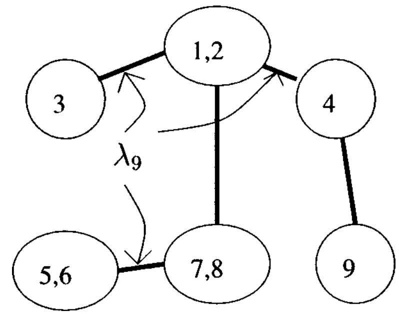

FIG. 8. A graphical representation of $[\lambda_9'',\theta_9]$ in (C17).

Using this relation in (C12), we finally obtain

$$b_N=\frac{1}{L}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}J([\lambda,\theta])S_\infty([\lambda,\theta]), \tag{C14}$$

which is (3.6). Note that the right-hand side of Eq. (C14) does not depend on L. Therefore, we do not write $\lim_{L\to\infty}$ any more.

In the no-interaction limit, where $(d\Delta/dk)(k)$ is replaced with $\delta(k)$, Eq. (C14) gives

$$\frac{1}{N}=\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\frac{1}{(N-1)!}$$
$$\times\left(\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)-1}(-1)^{(M_\sigma-1)}(M_\sigma-1)!\right). \tag{C15}$$

Next, we prove (3.9). We define

$$D([\lambda,\theta],\lambda')$$
$$=[\lambda'',\theta'']$$
$$\equiv\left[\left\{\{\sigma_1,\sigma_2\}\middle|\begin{array}{c}\sigma_1,\sigma_2\in g([\lambda',\theta])\\ \sigma_3\subset\sigma_1\ \ \sigma_4\subset\sigma_2\\ \{\sigma_3,\sigma_4\}\in\lambda,\notin\lambda'\end{array}\right\},g([\lambda',\theta])\right],$$
$$\lambda'\subseteq\lambda. \tag{C16}$$

θ'' is a set of a direct sum of elements linked with one cluster in λ', and λ'' is a connection pattern of a set θ''. A connection in λ'' links two elements of θ'', where elements of θ, each of which is a subset of one of the two elements, are connected by λ. For example,

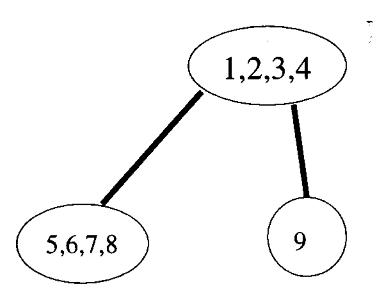

FIG. 9. A graphical representation of $[\lambda_9',\theta_9']$ in (C17).

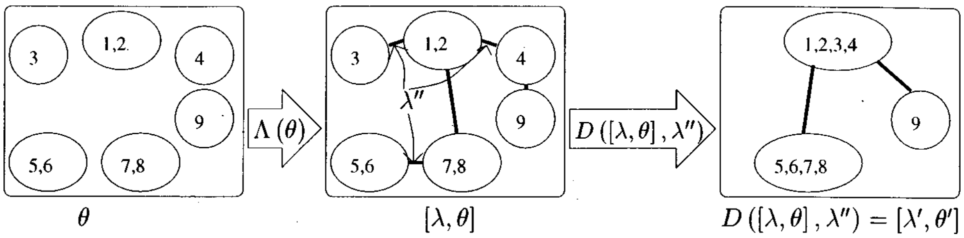

FIG. 10. A graphical representation of one of the patterns which are summed up in the left-hand side of Eq. (C24).

$$D([\lambda_9'',\theta_9],\lambda_9)=[\lambda_9',\theta_9'], \tag{C17}$$

where

$$\lambda_9=\{\{\{1,2\},\{3\}\},\{\{1,2\},\{4\}\},\{\{5,6\},\{7,8\}\}\}, \tag{C18}$$

$$\theta_9=\{\{1,2\},\{3\},\{4\},\{5,6\},\{7,8\},\{9\}\}, \tag{C19}$$

$$\lambda_9''=\left\{\begin{matrix}\{\{1,2\},\{3\}\},\{\{1,2\},\{4\}\},\{\{1,2\},\{7,8\}\},\\ \{\{4\},\{9\}\},\{\{5,6\},\{7,8\}\}\end{matrix}\right\}, \tag{C20}$$

$$\lambda_9'=\{\{\{1,2,3,4\},\{5,6,7,8\}\},\{\{1,2,3,4\},\{9\}\}\}, \tag{C21}$$

$$\theta_9'=\{\{1,2,3,4\},\{5,6,7,8\},\{9\}\}. \tag{C22}$$

Figures 8 and 9 illustratey $[\lambda_9'',\theta_9]$ and $[\lambda_9',\theta_9']$ in (C17).

By substituting $K(k)+2\pi\delta(k)$ into $(d\Delta/dk)(k)$, Eq. (C14) is changed into

$$b_N=\frac{1}{L}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}J([\lambda,\theta])\sum_{\lambda'\subseteq\lambda}S_\infty'(D([\lambda,\theta],\lambda')),$$

where $S_\infty'([\lambda,\theta])$ is defined in (3.11). With the explicit expression of $J([\lambda,\theta])$, we have

$$b_N=\frac{1}{(N-1)!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\sum_{\lambda'\subseteq\lambda}S_\infty'(D([\lambda,\theta],\lambda'))$$
$$\times\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)-1}(-1)^{(M_\sigma-1)}(M_\sigma-1)!. \tag{C23}$$

We can show that for arbitrary functions $f_1(n,\sigma)$ and $f_2([\lambda,\theta])$,

$$\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\sum_{\lambda''\subseteq\lambda}f_2(D([\lambda,\theta],\lambda''))\prod_{\sigma\in\theta}f_1(n(\sigma,\lambda),\sigma)$$
$$=\sum_{\theta'\in\Theta(N)}\sum_{\lambda'\in\Lambda_c(\theta')}f_2([\lambda',\theta'])$$
$$\times\prod_{\sigma'\epsilon\theta'}\sum_{\theta''\in\Theta(\sigma')}\sum_{\lambda''\epsilon\Lambda_c(\theta'')}\sum_{\sigma_1,\ldots,\sigma_{n(\sigma',\lambda')}\in\theta''}$$
$$\times\prod_{\sigma\epsilon\theta''}f_1\left(n(\sigma,\lambda'')+\sum_{i=1}^{n(\sigma',\lambda')}\delta_{\sigma,\sigma_i},\sigma\right). \tag{C24}$$

The meaning of the left-hand side of (C24) is to sum up all the patterns generated by the following process: first, divide a set into elements of θ; second, define a connection pattern λ; then make θ' and λ' join elements of θ together which are connected with λ''. The meaning of the right-hand side of (C24) is to sum up all the patterns generated by the following process: first, divide a set into elements of θ'; second, make a connection pattern λ'; third, divide each element of the set θ' into elements of θ''; fourth, define connection pattern λ'' on elements of θ'' each of which is a subset of an element in θ'; then connect elements each of which is a subset of one of two elements in θ' linked with a connection pattern λ'. Figures 10 and 11 show graphical representations of both sides of patterns in (C24). Equation (C24) is similar to (C11), in the sense that if $f_1(n,\sigma)$ and $f_2([\lambda,\theta])$ in Eq. (C24) are independent of n and λ, there is no connection pattern dependence, then Eq. (C24) becomes (C11).

By use of (C24), we rewrite (C23) as

$$b_N=\frac{1}{(N-1)!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}S_\infty'([\lambda,\theta])\prod_{\sigma\in\theta}\sum_{\theta'\in\Theta(\sigma)}\sum_{\lambda'\in\Lambda_c(\theta')}\sum_{\sigma_1',\ldots,\sigma'_{n(\lambda,\theta)}\in\theta'}\prod_{\sigma'\in\theta'}M_{\sigma'}^{n(\sigma',\lambda')+\Sigma_{i=1}^{n(\sigma,\lambda)}\delta_{\sigma',\sigma_i'}-1}(-1)^{M_{\sigma'}-1}$$
$$\times(M_{\sigma'}-1)!.$$

Since

$$\sum_{\sigma_1,\ldots,\sigma_n\in\theta}\prod_{\sigma\in\theta}M_\sigma^{\Sigma_{i=1}^n\delta_{\sigma,\sigma_i}}=\left(\sum_{\sigma\in\theta}M_\sigma\right)^n, \tag{C25}$$

b_N becomes

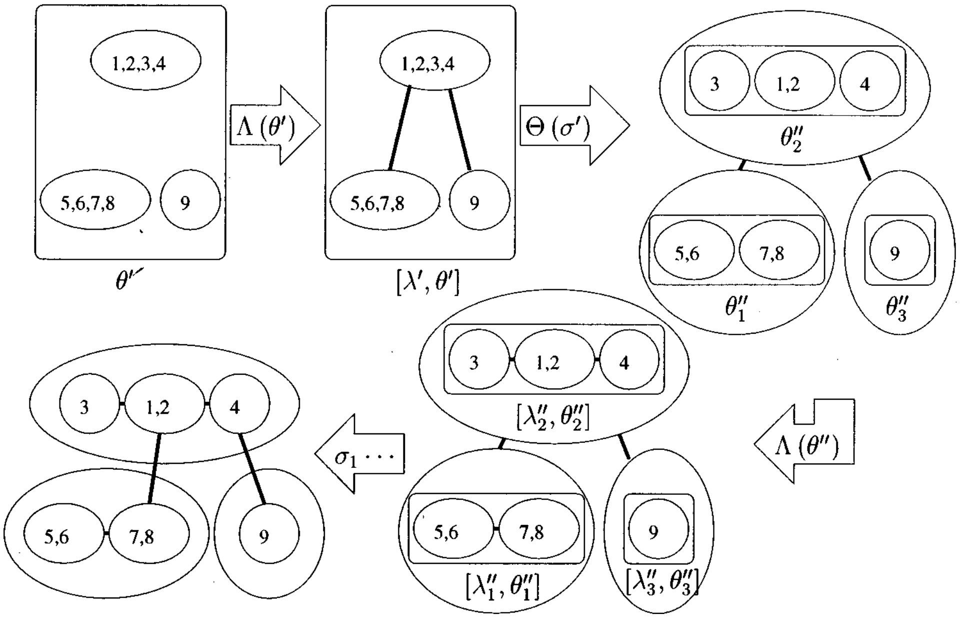

FIG. 11. A graphical representation of one of the patterns which are summed up in the right-hand side of Eq. (C24).

$$b_N=\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}S'_\infty([\lambda,\theta])\frac{1}{(N-1)!}\prod_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)}$$
$$\times\sum_{\theta'\in\Theta(\sigma)}\sum_{\lambda'\in\Lambda_c(\theta')}\left(\prod_{\sigma'\in\theta'}M_{\sigma'}^{n(\sigma',\lambda')-1}\right.$$
$$\left.\times(-1)^{M_{\sigma'}-1}(M_{\sigma'}-1)!\right). \quad \text{(C26)}$$

Using the relation (C15), we arrive at (3.9),

$$b_N=\frac{1}{L}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}J'([\lambda,\theta])S'_\infty([\lambda,\theta]), \quad \text{(C27)}$$

where $J'([\lambda,\theta])$ is defined in (3.10).

APPENDIX D: THE CLUSTER INTEGRAL (3.9) BY USE OF THE TBA

A result of the bosonic formulation of the TBA [5] is

$$p\beta=-\int\frac{dk}{2\pi}\log(1-z\exp(-\beta(k^2+\pi(k)))), \quad \text{(D1)}$$

where

$$\pi(k)=\frac{1}{\beta}\int\frac{dq}{2\pi}K(k-q)\log(1-z\exp(-\beta(q^2+\pi(q)))). \quad \text{(D2)}$$

We define a function

$$F_N(k)\equiv\frac{1}{N!}\frac{\partial^N\exp(-\beta\pi(k))}{\partial z^N}\bigg|_{z=0}, \quad \text{(D3)}$$

and prove that

$$F_N(k)=\frac{1}{N!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta_+)}\left(\prod_{\sigma\in\theta,\sigma\neq\{0\}}(M_\sigma-1)!\right.$$
$$\left.\times M_\sigma^{n(\sigma,\lambda)-1}\right)\tilde{S}([\lambda,\theta],k,\{0\}), \quad \text{(D4)}$$

$$F_0(k)=1, \quad \text{(D5)}$$

where for $\sigma\in\theta,\theta_+\equiv\theta\cup\{\{0\}\}$,

$$\tilde{S}([\lambda,\theta],k_\sigma,\sigma)$$
$$\equiv\int\prod_{\sigma'\in\theta,\sigma'\neq\sigma}\frac{dk_{\sigma'}}{2\pi}\left(\prod_{\{\sigma',\sigma''\}\in\lambda}K(k_{\sigma'}-k_{\sigma''})\right)$$
$$\times\exp\left(-\beta\sum_{\sigma'\in\theta,\sigma'\neq\sigma}M_{\sigma'}k_{\sigma'}^2\right). \quad \text{(D6)}$$

It can be easily checked that (D5) is true.

From Eq. (D2), a recursive relation for $F_N(k)$ is obtained as

$$F_N(k)=\frac{1}{N!}\sum_{\theta\in\Theta(N)}\prod_{\sigma\in\theta}\int\frac{dq_\sigma}{2\pi}K(k-q_\sigma)$$
$$\times\sum_{\theta'\in\Theta(M_\sigma)}(M_{\theta'}-1)!\exp(-\beta M_{\theta'}q_\sigma^2)$$
$$\times\prod_{\sigma'\in\theta'}M_{\sigma'}!F_{M_{\sigma'}-1}(q_\sigma). \quad \text{(D7)}$$

Substituting the expression of $F_N(k)$ in (D4) into $F_{M_\sigma-1}(q)$ in (D7), we obtain

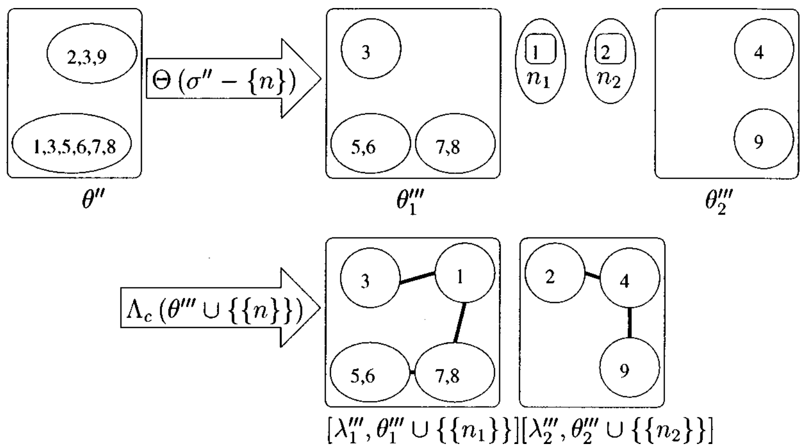

FIG. 12. A graphical representation of one of the patterns which are summed up in the left-hand side of Eq. (D10).

$$F_N(k)=\frac{1}{N!}\sum_{\theta\in\Theta(N)}\prod_{\sigma\in\theta}\int\frac{dq_\sigma}{2\pi}K(k-q_\sigma)\sum_{\theta'\in\Theta(M_\sigma)}(M_{\theta'}-1)!\exp(-\beta M_{\theta'}q_\sigma^2)$$
$$\times\prod_{\sigma'\in\theta'}M_{\sigma'}\sum_{\theta''\in\Theta(M_{\sigma'}-1)}\sum_{\lambda''\in\Lambda_c(\theta''_+)}\left(\prod_{\sigma''\in\theta'',\neq\{0\}}(M_{\sigma''}-1)!M_{\sigma''}^{n(\sigma'',\lambda'')-1}\right)\tilde{S}([\lambda'',\theta''_+],q_\sigma,\{0\}). \tag{D8}$$

For noninteger argument of $\Theta(\sigma)$, (D8) becomes

$$F_N(k)=\frac{1}{N!}\sum_{\theta\in\Theta(N)}\prod_{\sigma\in\theta}\int\frac{dq_\sigma}{2\pi}K(k-q_\sigma)\sum_{\theta'\in\Theta(\sigma)}(M_{\theta'}-1)!\exp(-\beta M_{\theta'}q_\sigma^2)$$
$$\times\prod_{\sigma'\in\theta'}\sum_{n\in\sigma}\sum_{\theta''\in\Theta(\sigma'-\{n\})}\sum_{\lambda''\in\Lambda_c(\theta''\cup\{\{n\}\})}\left(\prod_{\sigma''\in\theta'',\neq\{n\}}(M_{\sigma''}-1)!M_{\sigma''}^{n(\sigma'',\lambda'')-1}\right)\tilde{S}([\lambda'',\theta''\cup\{\{n\}\}],q_\sigma,\{n\}). \tag{D9}$$

We can show that

$$\sum_{\theta''\in\Theta(N)}f_1(M_{\theta''})\prod_{\sigma''\in\theta''}\sum_{n\in\sigma''}\sum_{\theta'''\in\Theta(\sigma''-\{n\})}\sum_{\lambda'''\in\Lambda_c(\theta'''\cup\{\{n\}\})}f_2([\lambda''',\theta'''\cup\{\{n\}\}],\{n\})\prod_{\sigma'\in\theta''',\neq\{n\}}f_3(n(\sigma',\lambda'''),\sigma')$$
$$=\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\sum_{\sigma\in\theta}M_\sigma^{n(\sigma,\lambda)}f_1(M_\sigma)f_2([\lambda,\theta],\sigma)\prod_{\sigma'\in\theta,\neq\sigma}f_3(n(\sigma',\lambda),\sigma'), \tag{D10}$$

where $f_1(n)$ and $f_3(n,\sigma)$ are arbitrary functions, and $f_2([\lambda,\theta],\sigma)$ is a function which satisfies

$$f_2([\lambda,\theta],\sigma)=\prod_i f_2([\lambda_i,\theta_i],\sigma_i), \tag{D11}$$

$$\lambda=\bigcup_i\lambda_i,\quad \theta=\bigcup_i(\theta_i-\{\sigma_i\})\cup\{\sigma\},\quad \sigma=\bigcup_i\sigma_i,$$

$$\sigma\in\theta,\quad \sigma_i\in\theta_i.$$

The meaning of the right-hand side of (D10) is to sum up all the patterns generated by the following process: first, divide a set $\{1,...,N\}$ into sets of θ; second define a connection pattern λ which consists of one cluster; then choose one set σ in θ. The meaning of the left-hand side of (D10) is to sum up all the patterns generated by the following process: first, di-

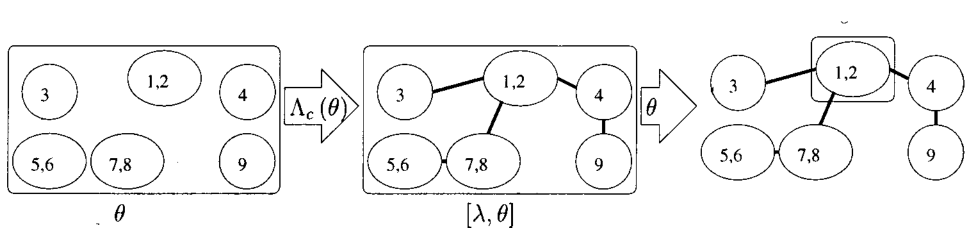

FIG. 13. A graphical representation of one of the patterns which are summed up in the right-hand side of Eq. (D10).

vide $\{1,...,N\}$ into sets of θ'' each of which contain one element n in the σ chosen in the right-hand side and elements of which some sets in θ consists. The σ chosen in the right-hand side is not one of these sets, and these sets are connected with each other by connections in λ and any sets are connected with the σ by connections in λ. Second, divide each of sets in θ'' into sets of θ' and $\{\{n\}\}$. Then, define a connection pattern λ'' of sets in θ''' and $\{\{n\}\}$. The connection pattern λ'' consists of one cluster. Figures 12 and 13 illustrate graphical representations of both sides of patterns in (D10). Note that if one chooses a right-hand side pattern, there exist the corresponding $M_\sigma^{n(\sigma,\lambda)}$ left-hand side patterns.

Applying the relation (D10) to (D9), we have

$$
\begin{aligned}
F_N(k)=&\frac{1}{N!}\sum_{\theta\in\Theta(N)}\prod_{\sigma\in\theta}\int\frac{dq_\sigma}{2\pi}K(k-q_\sigma)\sum_{\theta'\in\Theta(\sigma)}\sum_{\lambda'\in\Lambda_c(\theta')}\sum_{\sigma'\in\theta'}M_{\sigma'}^{n(\sigma',\lambda')}(M_{\sigma'}-1)!\exp(-\beta M_{\sigma'}q_\sigma^2)\\
&\times\tilde{S}([\lambda',\theta'],q_\sigma,\sigma')\prod_{\sigma''\in\theta',\sigma''\neq\sigma'}(M_{\sigma''}-1)!M_{\sigma''}^{n(\sigma'',\lambda')-1}\\
=&\frac{1}{N!}\sum_{\theta\in\Theta(N)}\prod_{\sigma\in\theta}\sum_{\theta'\in\Theta(\sigma)}\sum_{\lambda'\in\Lambda_c(\theta')}\sum_{\sigma'\in\theta'}\int\frac{dq_{\sigma'}}{2\pi}K(k-q_{\sigma'})M_{\sigma'}^{n(\sigma',\lambda')}(M_{\sigma'}-1)!\\
&\times\exp(-\beta M_{\sigma'}q_{\sigma'}^2)\tilde{S}([\lambda',\theta'],q_{\sigma'},\sigma')\prod_{\sigma''\in\theta',\sigma''\neq\sigma'}(M_{\sigma''}-1)!M_{\sigma''}^{n(\sigma'',\lambda')-1}\\
=&\frac{1}{N!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda(\theta)}\sum_{[\lambda',\theta']\in\tilde{G}([\lambda,\theta])}\sum_{\sigma'\in\theta'}\int\frac{dq_{\sigma'}}{2\pi}K(k-q_{\sigma'})M_{\sigma'}^{n(\sigma',\lambda')}(M_{\sigma'}-1)!\\
&\times\exp(-\beta M_{\sigma'}q_{\sigma'}^2)\tilde{S}([\lambda',\theta'],q_{\sigma'},\sigma')\prod_{\sigma''\in\theta',\sigma''\neq\sigma'}(M_{\sigma''}-1)!M_{\sigma''}^{n(\sigma'',\lambda')-1}.
\end{aligned}\tag{D12}
$$

The second equality is due to a replacement of integration variables from q_σ to $q_{\sigma'}$, and the relation (C7) is used in the third equality.

By definitions, the following relation can be shown:

$$
\begin{aligned}
\tilde{S}([\lambda',\theta_+],k,\{0\})=&\prod_{[\lambda_i,\theta_i]\in\tilde{G}([\lambda,\theta])}\int\frac{dq_{\sigma_i}}{2\pi}K(k-q_{\sigma_i})\\
&\times\exp(-\beta M_{\sigma_i}q_{\sigma_i}^2)\tilde{S}([\lambda_i,\theta_i],q_{\sigma_i},\sigma_i),
\end{aligned}
$$

$$
\lambda'\in\Lambda_c(\theta_+),\quad\lambda\in\Lambda(\theta),\quad\sigma_i\in\theta_i,\quad\lambda'=\lambda\cup\bigcup_i\{\sigma_i,\{0\}\}.\tag{D13}
$$

Using relation (D13) in (D12), we obtain

$$
\begin{aligned}
F_N(k)=&\frac{1}{N!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta_+)}\left(\prod_{\sigma\in\theta,\sigma\neq\{0\}}(M_\sigma-1)!\right.\\
&\left.\times M_\sigma^{n(\sigma,\lambda)-1}\right)\tilde{S}([\lambda,\theta_+],k,\{0\}).
\end{aligned}\tag{D14}
$$

We see that (D14) is equal to (D4). Therefore, Eq. (D4) is recursively proved.

From (D1), we have

$$
\begin{aligned}
b_N=&\int\frac{dk}{2\pi}\sum_{\theta\in\Theta(N)}\exp(-M_\theta\beta k^2)\frac{(M_\theta-1)!}{N!}\\
&\times\prod_{\sigma\in\theta}M_\sigma!F_{M_\sigma-1}(k).
\end{aligned}\tag{D15}
$$

Substituting the expression of F_N in (D4) into $F_{M_\sigma-1}$ in (D15) gives

$$
\begin{aligned}
b_N=&\int\frac{dk}{2\pi}\sum_{\theta\in\Theta(N)}\exp(-M_\theta\beta k^2)\frac{(M_\theta-1)!}{N!}\\
&\times\prod_{\sigma\in\theta}M_\sigma\sum_{\theta'\in\Theta(M_\sigma-1)}\sum_{\lambda'\in\Lambda_c(\theta'_+)}\\
&\times\left(\prod_{\sigma'\in\theta',\sigma'\neq\{0\}}(M_{\sigma'}-1)!M_{\sigma'}^{n(\sigma',\lambda')-1}\right)\\
&\times\tilde{S}([\lambda',\theta'_+],k,\{0\}).
\end{aligned}\tag{D16}
$$

Using the relation (D10), we have

$$b_N=\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\sum_{\sigma\in\theta}\int\frac{dk}{2\pi}\exp(-M_\sigma\beta k^2)\times M_\sigma^{n(\sigma,\lambda)}\frac{(M_\sigma-1)!}{N!}\tilde{S}([\lambda,\theta],k,\sigma)\times\prod_{\sigma'\in\theta,\sigma'\neq\sigma}(M_{\sigma'}-1)!M_{\sigma'}^{n(\sigma',\lambda)-1}. \tag{D17}$$

Since the relation

$$S'_\infty([\lambda,\theta])=\int\frac{dq_\sigma}{2\pi}\exp(-\beta M_\sigma q_\sigma^2)\tilde{S}([\lambda,\theta],q_\sigma,\sigma),\quad \lambda\in\Lambda_c(\theta),\quad \sigma\in\theta, \tag{D18}$$

holds, we arrive at

$$b_N=\frac{1}{N!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda_c(\theta)}\left(\sum_{\sigma\in\theta}M_\sigma\right)\times\left(\prod_{\sigma\in\theta}(M_\sigma-1)!M_\sigma^{n(\sigma,\lambda)-1}\right)S'_\infty([\lambda,\theta]) \tag{D19}$$

$$=\frac{1}{(N-1)!}\sum_{\theta\in\Theta(N)}\sum_{\lambda\in\Lambda(\theta)}\left(\prod_{\sigma\in\theta}(M_\sigma-1)!\times M_\sigma^{n(\sigma,\lambda)-1}\right)S'_\infty([\lambda,\theta]), \tag{D20}$$

where $S'_\infty(\lambda,\theta)$ is defined in (3.11). This expression of the cluster integral b_N is the same as (3.9).

[1] E. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[2] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[3] H. A. Bethe, Z. Phys. **71**, 205 (1931).
[4] H. B. Thacker, Phys. Rev. D **16**, 2515 (1977).
[5] M. Wadati, J. Phys. Soc. Jpn. **54**, 3727 (1985).
[6] G. Kato and M. Wadati, Chaos, Solitons Fractals (to be published).